IEOR 8100-001: Learning and Optimization for Sequential Decision Making

Homework 1

Instructor: Shipra Agrawal **Due on: Feb 10, 2016**

Note. *This homework is meant for assessing your mathematical background for the course. This will be graded pass/fail. You are not expected to solve all the problems, some of them can be quite difficult while others should be readily solvable if you have mastered the prerequisites for the course. You must work on this assignment by yourself. Turn in the solutions in class or email them to ieor8100-01-spring2016@columbia.edu before class, on the due date.*

Problem 1. Prove that $e^{\sqrt{\ln(r)}} = o(r)$. Here $\ln(\cdot)$ denotes logarithm with base e.

Problem 2. There is 75% chance of rain on Monday. There is 50% chance of rain on Tuesday if that it rains on Monday, and 20% otherwise. If it rains on any given day, there is a 70% chance that Alice will get wet on her way to work. (She does not change her behavior based on what happened the earlier day). What is the probability that Alice will get wet from rain on both Monday and Tuesday?

Problem 3. A standard 52-card deck [1] is randomly partitioned into four 13-element sets, which are dealt to players named Alice, Bob, Tom, and Harry.

(a) Calculate Pr(Alice gets exactly 2 aces|Bob gets exactly 1 ace).
Hint: Use $\Pr(A|B) = \Pr(A \cap B)/\Pr(B)$.

(b) Let C and S denote the number of clubs and spades, respectively, dealt to Alice. Calculate $E(C|S)$ as a function of S.
Hint : Observe that given the number of spades, the number of clubs, hearts, diamonds dealt to Alice have the same conditional distribution.

Problem 4 Let $X_1, X_2, \ldots, X_n$ be independent uniformly-distributed random samples from the interval $[0, 1]$. Define the following probabilities:

- $p(n)$ is the probability that $\min_k X_k > 0.01$.
- $q(n)$ is the probability that $\min_{i \neq j} |X_i - X_j| \leq \frac{1}{n^2}$.
- $r(n)$ is the probability that $\min_{i \neq j} |X_i - X_j| > \frac{1}{100n}$.
- $s(n)$ is the probability that exactly $\lfloor n/2 \rfloor$ of the numbers $X_1, ..., X_n$ lie in $[0, \frac{1}{2}]$.

Estimate the asymptotic behavior of each of these probabilities as n tends to infinity. Specifically, for each of the sequences $p(n), q(n), r(n), s(n)$, determine whether the sequence

(A) tends to zero exponentially fast, i.e. is $O(c^n)$ for some constant $c < 1$;

(B) tends to zero, but not exponentially fast;

(C) remains bounded away from 0 and 1;

(D) tends to 1.

You do not need to justify your answer. Just answer A, B, C, or D for each of the sequences.

[1] A standard 52-card deck is the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, ace\} \times \{clubs, diamonds, hearts, spades\}$.

Problem 5

1. (a) You are selling your bike. You get offers one by one. You have decided to stop and accept an offer as soon as you see one which is better than the first offer you got. (You always skip the first offer). What is the expected number of offers you will to wait for, including the first one, until you accept an offer? Mathematically, let's model this process as follows. Let X_1, X_2, ... denote an infinite sequence of independent uniformly-distributed random samples from the interval $[0, 1]$. (Interpretation: X_i is the i^{th} offer.) Let τ be the smallest $i > 1$ such that $X_i > X_1$. What is $E[\tau]$?

2. (b) Now suppose that you modify your stopping rule. For some fixed predetermined number k, you do not accept any of the first k offers. Let h be the second best offer observed among the first k. Your policy is to accept the next offer (after the first k) which is better than h. In more precise terms, let X_1, X_2, . . . be a sequence of independent random variables uniformly distributed in $[0, 1]$ as before, let X_a, X_b be the two largest elements of the set $X_1, \ldots, X_k$, and let ρ be the smallest $i > k$ such that $X_i > X_b$. What is $E(\rho)$, as a function of k?

Hint: Use following formula for expected value of a non-negative integer valued random variable Y: $E[Y] = \sum_{n=0}^{\infty} \Pr(Y > n)$.